Factorization of formal exponentials and uniformization

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Abstract

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero equipped with a vector space decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^+$, and let s and t be commuting formal variables commuting with $\mathfrak{g}$. We prove that the map $C : s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \oplus t\mathfrak{g}^+[[s,t]]$ defined by the Campbell-Baker-Hausdorff formula and given by $e^{sg^-}e^{tg^+} = e^{C(sg^-,tg^+)}$ for $g^\pm \in \mathfrak{g}^\pm[[s,t]]$ is a bijection, as is well known when $\mathfrak{g}$ is finite-dimensional over $\mathbb{R}$ or $\mathbb{C}$, by geometry. It follows that there exist unique $\Psi^\pm \in \mathfrak{g}^\pm[[s,t]]$ such that $e^{tg^+}e^{sg^-} = e^{s\Psi^-}e^{t\Psi^+}$ (also well known in the finite-dimensional geometric setting). We apply this to a Lie algebra $\mathfrak{g}$ consisting of certain formal infinite series with coefficients in a $\mathbb{Z}$-graded Lie algebra $\mathfrak{p}$, for instance, an affine Lie algebra, the Virasoro algebra or a Grassmann envelope of the $N = 1$ Neveu-Schwarz superalgebra. For $\mathfrak{p}$ the Virasoro algebra, the result was first proved by Huang as a step in the construction of a geometric formulation of the notion of vertex operator algebra, and for $\mathfrak{p}$ a Grassmann envelope of the Neveu-Schwarz superalgebra, it was first proved by Barron as a corresponding step in the construction of a supergeometric formulation of the notion of vertex operator superalgebra. In the special case of the Virasoro (resp., $N = 1$ Neveu-Schwarz) algebra with zero central charge the result gives the precise expansion of the uniformizing

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function for a sphere (resp., supersphere) with tubes resulting from the sewing of two spheres (resp., superspheres) with tubes in two-dimensional genus-zero holomorphic conformal (resp., $N = 1$ superconformal) field theory. The general result places such uniformization problems into a broad formal algebraic context.

1 Introduction

Recall that for a Lie algebra $\mathfrak{g}$ over a field of characteristic zero and $g_1, g_2 \in \mathfrak{g}$, the classical Campbell-Baker-Hausdorff formula (cf. [R]) gives a formal Lie series $C(g_1, g_2)$ in g_1 and g_2 such that $e^{g_1}e^{g_2} = e^{C(g_1,g_2)}$. In this paper we prove the following factorization theorem (see Theorem 3.1 below): Let $\mathfrak{g}$ be equipped with a vector space decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^+$. Then the map

$$C : s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \oplus t\mathfrak{g}^+[[s,t]] \tag{1}$$

defined by the Campbell-Baker-Hausdorff formula and given by

$$e^{sg^-}e^{tg^+} = e^{C(sg^-,tg^+)} \tag{2}$$

for $g^{\pm} \in \mathfrak{g}^{\pm}[[s,t]]$ is a bijection; here s and t are commuting formal variables commuting with $\mathfrak{g}$. The map C is just the Campbell-Baker-Hausdorff formula inside the Lie algebra $\mathfrak{g}[[s,t]]$, the coefficients s and t in (2) making this possibly infinite Lie series well-defined. The content of our theorem is that C^{-1} exists and is a factorization of formal exponentials with respect to this vector space decomposition. (Note that in the domain of the map C in (1), we have the cartesian product of the two spaces, while in the codomain we of course have the same set but viewed as the vector space direct sum.)

In the case that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, this result is well known and is proved using the geometry of a corresponding Lie group (see e.g. [V]). But in the case of infinite-dimensional Lie algebras, since the theory of the corresponding group-like structures and of the correspondence between Lie algebras and these group-like structures is not developed, the argument proving in the finite-dimensional Lie algebra case that for given $h^{\pm} \in \mathfrak{g}^{\pm}[[s,t]]$, one can factor $e^{sh^- + th^+}$ uniquely as $e^{sg^-}e^{tg^+}$ for some $g^{\pm} \in \mathfrak{g}^{\pm}[[s,t]]$ cannot be generalized directly. Even more to the point, in the case of the Virasoro algebra and the $N = 1$ Neveu-Schwarz (superconformal) algebra, special cases of this result were indeed needed and proved

in the very study of the correspondence between the infinite-dimensional Lie algebras and the group-like structures (see [H1]-[H3] and [B1]-[B2]). To our knowledge, the present general formal factorization result in the infinite-dimensional case has not previously appeared in the literature. Regardless, in the present work we are mainly concerned with applications of the result (in the infinite-dimensional case).

We also give precise information about the form of the elements sg^- and tg^+ in terms of sh^- and th^+ by defining universal formal series that we call the "canonical factorization series" $F^\pm$, which are certain formal infinite linear combinations of "words" in sh^-, th^+ and the canonical projections $\pi^\pm : \mathfrak{g}[[s,t]] \to \mathfrak{g}^\pm[[s,t]]$, and showing that $sg^- = F^-(sh^-, th^+; \pi^\pm)$ and $tg^+ = F^+(sh^-, th^+; \pi^\pm)$.

As a corollary of the factorization theorem stated above, we use (1) and (2) to construct (see Corollary 3.12 below) a unique bijection:

$$\Psi : t\mathfrak{g}^+[[s,t]] \times s\mathfrak{g}^-[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]] \tag{3}$$

such that for $g^\pm \in \mathfrak{g}^\pm[[s,t]]$,

$$e^{tg^+}e^{sg^-} = e^{s\Psi^-}e^{t\Psi^+} \tag{4}$$

with $s\Psi^- = \pi^- \circ \Psi(tg^+, sg^-)$ and $t\Psi^+ = \pi^+ \circ \Psi(tg^+, sg^-)$. We call this result "formal algebraic uniformization," for reasons about to be explained. It follows from the information contained in the canonical factorization series arising from the factorization theorem, and the Campbell-Baker-Hausdorff theorem, that $s\Psi^-$ and $t\Psi^+$ are equal to certain universal formal series of words in sg^-, tg^+ and the projections $\pi^\pm$. We call these canonical series the "formal algebraic uniformization series."

In our applications, which we are about to describe, we use the existence of the canonical factorization series and of the formal algebraic uniformization series as steps in the proof. These formal series of words can be thought of as analogues, in some sense, of the Campbell-Baker-Hausdorff series.

For $\mathfrak{p}$ a $\mathbb{Z}$-graded Lie algebra, we apply the formal algebraic uniformization (and also factorization) results to a Lie subalgebra $\mathfrak{g}$ of the Lie algebra consisting of certain formal infinite series with coefficients in $\mathfrak{p}$ (Corollary 4.1 below). In [H1], Huang proved Corollary 4.1 in the case where $\mathfrak{p}$ is the Virasoro algebra (see Application 4.3 below) by first establishing the result in a certain representation of the Virasoro algebra, namely, the standard representation given by $L_n \mapsto -x^{n+1}\frac{d}{dx} \in \mathrm{End}(\mathbb{C}[x, x^{-1}])$ and $c = 0$, and then

lifting to representations with general c, and extending to a modification of the universal enveloping algebra. In [B1], Barron used a similar approach to prove Corollary 4.1 in the case where $\mathfrak{p}$ is a Grassmann envelope of the $N = 1$ Neveu-Schwarz superalgebra (see Application 4.7 below). These two cases are fundamental to the sewing operations in conformal and $N = 1$ superconformal field theory, respectively (see [H1]-[H3], [B1]-[B2]). In the case of the Virasoro (resp., $N = 1$ Neveu-Schwarz) algebra with zero central charge the result gives the precise expansion of the uniformizing function for a sphere (resp., supersphere) with tubes resulting from the sewing of two spheres (resp., superspheres) with tubes in two-dimensional genus-zero holomorphic conformal (resp., $N = 1$ superconformal) field theory. This paper gives a unified proof of these two results and further shows that these are special cases of a much more general result, namely, Corollary 4.1. This corollary can in addition be applied to obtain the corresponding results for a Grassmann envelope of the $N > 1$ Neveu-Schwarz algebras of supersymmetries in $N > 1$ superconformal field theories, and also to such structures as affine Lie algebras (see Application 4.2 below) and superalgebras. Since Corollary 4.1 is in turn a special case of Corollary 3.12, we can perhaps view Corollary 3.12 as a generalized and canonical formal algebraic version of such uniformization.

To prove the results of the present paper, one might have hoped that the method used in [H1] and [B1] to obtain the results in special representations could be generalized directly to the universal enveloping algebra arising in the formulations of our main results. However, the direct generalization of that method to the universal enveloping algebra does not work because the method in [H1] and [B1] uses properties of the special representations that universal enveloping algebras do not have. In the present paper, instead of working in the universal enveloping algebra, we work directly in the Lie algebra (see Remark 3.10 below). In particular, we reformulate the desired results as equations in the Lie algebra, and the method in [H1] and [B1] used for special representations can now be applied in the Lie algebra to solve these equations and thus obtain the results (see Remark 3.11 below). In fact a crucial observation in this paper is that even though there is no setting involving universal enveloping algebras to which the method of proof used in [H1] and [B1] can be applied, one can in fact still find a very general setting to which the method can be applied, a setting very different from the ones in [H1] and [B1].

This paper is organized as follows. In Section 2, we give a review of

the Campbell-Baker-Hausdorff formula, including some notation that will be useful later. In Section 3, we prove the main theorem on the factorization of formal exponentials, we establish a number of related results, and we use the main theorem to prove the corollary giving formal algebraic uniformization. We also introduce the canonical factorization series and the formal algebraic uniformization series. In Section 4, we apply these results to Lie algebras consisting of certain formal infinite series with coefficients in a $\mathbb{Z}$-graded Lie algebra. We then give several applications of the result for affine Lie algebras, the Virasoro algebra and Grassmann envelopes of the $N = 1$ Neveu-Schwarz superalgebra. For the latter two applications, we discuss the importance of these results to conformal and superconformal field theory, and we point out that the result also applies to Grassmann envelopes of the $N > 1$ Neveu-Schwarz algebras.

2 The Campbell-Baker-Hausdorff formula

We begin with some review of the Campbell-Baker-Hausdorff formula, following [R]. Let $\mathbb{F}$ be a field of characteristic zero, and let a and b be two formal noncommuting symbols. Let $\mathbb{F}\langle a, b\rangle$ be the $\mathbb{F}$-algebra of formal linear combinations of words in a and b, i.e., noncommutative polynomials in a and b over $\mathbb{F}$. A *Lie polynomial* in a and b is an element of the smallest $\mathbb{F}$-subspace of $\mathbb{F}\langle a, b\rangle$ containing a and b and closed under Lie brackets. For an element S of the formal completion $\mathbb{F}\langle\langle a, b\rangle\rangle$ of $\mathbb{F}\langle a, b\rangle$, write

$$S = \sum_{n\in\mathbb{N}} S_n,$$

where each S_n is homogeneous of total degree n in a and b. Then S is called a *Lie series* if each S_n is a Lie polynomial. For any formal series $S \in \mathbb{F}\langle\langle a, b\rangle\rangle$ with no constant term, its formal exponential

$$e^S = \sum_{n\in\mathbb{N}} \frac{S^n}{n!}$$

is well defined.

The Campbell-Baker-Hausdorff theorem asserts the existence of a (unique) Lie series $C(a, b) \in \mathbb{Q}\langle\langle a, b\rangle\rangle$ such that

$$e^a e^b = e^{C(a,b)}. \tag{5}$$

We now recall the precise formula for $C(a,b)$. Even though we will not need the main information contained in it, it is nice to see the role that it plays in our proof of the bijectivity of C in Theorem 3.1 below. For $S \in \mathbb{Q}\langle\langle a,b\rangle\rangle$, let $S\frac{\partial}{\partial b}$ denote the derivation of $\mathbb{Q}\langle\langle a,b\rangle\rangle$ that maps a to 0 and b to S. The series $C(a,b)$ is given by

$$C(a,b) = \exp\left(H_1\frac{\partial}{\partial b}\right)(b), \tag{6}$$

where

$$H_1 = \left(\frac{\mathrm{ad}\ b}{e^{\mathrm{ad}\ b}-1}\right)(a),$$

and H_1 is the part of $C(a,b)$ that is homogeneous of degree one with respect to a. In particular, if we write

$$C(a,b) = \sum_{j\in\mathbb{N}} H_j \tag{7}$$

where H_j is the part of $C(a,b)$ that is homogeneous of degree j with respect to a, then

$$H_j = \frac{1}{j!}\left(H_1\frac{\partial}{\partial b}\right)^j(b)$$

(cf. [R]). Furthermore, writing

$$H_j = \sum_{k\in\mathbb{N}} H_{j,k}, \tag{8}$$

where $H_{j,k} \in \mathbb{Q}\langle a,b\rangle$ is homogeneous of degree j in a and degree k in b, we note that

$$H_{j,0} = \begin{cases} a & \text{if } j=1 \\ 0 & \text{otherwise} \end{cases}, \qquad H_{0,k} = \begin{cases} b & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

and

$$H_{1,1} = \frac{1}{2}[a,b]. \tag{10}$$

3 Factorization of formal exponentials and formal algebraic uniformization

In this section we state and prove the two main results of the paper: factorization of formal exponentials (Theorem 3.1) and formal algebraic uniformization (Corollary 3.12). We also give precise information about the form of the resulting elements (Theorem 3.6 and Corollary 3.13).

We work over our field $\mathbb{F}$ of characteristic zero. We fix a Lie algebra $\mathfrak{g}$. We want to use the Campbell-Baker-Hausdorff formula, where the formal symbols a and b in the series $C(a,b)$ are now replaced by elements of $\mathfrak{g}$, and we want to consider the resulting Lie series as an element of $\mathfrak{g}$ rather than as a formal series. In other words, we want to evaluate the brackets within the Lie algebra $\mathfrak{g}$, but in general, the series might not be well defined in $\mathfrak{g}$ since it will typically contain infinitely many nonzero terms. However, let us introduce commuting formal variables s and t commuting with $\mathfrak{g}$. We consider the Campbell-Baker-Hausdorff series $C(sg_1, tg_2)$ for any $g_1, g_2 \in \mathfrak{g}$. From (6), (7) and (8), we see that each $H_{j,k}$ for $j,k \in \mathbb{N}$ in the Lie series $C(sg_1, tg_2)$ involves only finitely many brackets in $\mathfrak{g}[[s,t]]$, and therefore when we evaluate brackets in $\mathfrak{g}$, the coefficient of a given $s^j t^k$ in $C(sg_1, tg_2)$ is well defined in $\mathfrak{g}$. Thus $C(sg_1, tg_2)$ is a well defined element of $\mathfrak{g}[[s,t]]$. Furthermore, note that if, more generally, $g_1, g_2 \in \mathfrak{g}[[s,t]]$, then for $j,k \in \mathbb{N}$, each $H_{j,k}$ in the Lie series $C(sg_1, tg_2)$ is a sum of terms of degree greater than or equal to j in s and k in t, so that $C(sg_1, tg_2)$ is still well defined in $\mathfrak{g}[[s,t]]$. Of course the exponentials e^{sg_1}, e^{tg_2} and $e^{C(sg_1,tg_2)}$ are elements of the algebra $U(\mathfrak{g})[[s,t]]$ of formal power series over the universal enveloping algebra $U(\mathfrak{g})$.

Fix a vector space decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^+, \tag{11}$$

so that $\mathfrak{g}[[s,t]] = \mathfrak{g}^-[[s,t]] \oplus \mathfrak{g}^+[[s,t]]$. Let

$$\pi^\pm : \mathfrak{g}[[s,t]] \longrightarrow \mathfrak{g}^\pm[[s,t]]$$

be the corresponding projection maps. Then we have:

Theorem 3.1 (The Factorization Theorem) *The map*

$$C : s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \oplus t\mathfrak{g}^+[[s,t]] \tag{12}$$

defined by the Campbell-Baker-Hausdorff formula and given by

$$e^{sg^-}e^{tg^+} = e^{C(sg^-,tg^+)} \tag{13}$$

for $g^\pm \in \mathfrak{g}^\pm[[s,t]]$ *is a bijection. The lowest order terms of* $C(sg^-, tg^+)$ *are described as follows:*

$$C(sg^-, tg^+) = sg^- + tg^+ + st\frac{1}{2}[g^-, g^+] + stq(s,t), \tag{14}$$

where $q(s,t) \in \mathfrak{g}[[s,t]]$ *and* $q(0,0) = 0$*. Moreover, for* $h^\pm \in \mathfrak{g}^\pm[[s,t]]$*, the lowest order terms of*

$$C^{-1}(sh^- + th^+) = (sg^-, tg^+)$$

are described as follows:

$$g^- = h^- + \frac{1}{2}t\pi^-([h^+, h^-]) + tr^-(s,t), \tag{15}$$

$$g^+ = h^+ + \frac{1}{2}s\pi^+([h^+, h^-]) + sr^+(s,t), \tag{16}$$

where $r^\pm(s,t) \in \mathfrak{g}^\pm[[s,t]]$ *and* $r^\pm(0,0) = 0$*.*

Proof: For $g^\pm \in \mathfrak{g}^\pm[[s,t]]$, the expression $e^{sg^-}e^{tg^+}$ is well defined in $U(\mathfrak{g})[[s,t]]$, and by the Campbell-Baker-Hausdorff theorem and the discussion above, there exists a unique element $C(sg^-, tg^+) \in \mathfrak{g}[[s,t]]$ such that (13) holds; moreover, (14) also holds.

For $g^\pm \in \mathfrak{g}^\pm[[s,t]]$, write $g^\pm = \sum_{m\in\mathbb{N}} g^\pm_m$ where $g^\pm_m$ is homogeneous of degree m in s, and write

$$g^\pm_m = \sum_{n\in\mathbb{N}} g^\pm_{m,n} \tag{17}$$

where $g^\pm_{m,n} \in \mathfrak{g}^\pm[s,t]$ is homogeneous of degree m in s and degree n in t, i.e., $g^\pm_{m,n}s^{-m}t^{-n} \in \mathfrak{g}^\pm$. To prove that C is bijective, given $h^\pm \in \mathfrak{g}^\pm[[s,t]]$, we will use recursion on m and n to construct unique $g^\pm \in \mathfrak{g}^\pm[[s,t]]$ such that

$$C(sg^-, tg^+) = sh^- + th^+. \tag{18}$$

We will use the notation $h^\pm_m$ and $h^\pm_{m,n}$, by analogy with (17).

For $g^{\pm} \in \mathfrak{g}^{\pm}[[s,t]]$, consider the series in $\mathfrak{g}[[s,t]]$ given by the Campbell-Baker-Hausdorff formula

$$C(sg^-, tg^+) = \sum_{j,k\in\mathbb{N}} H_{j,k}(sg^-, tg^+) \tag{19}$$

where $H_{j,k}(sg^-, tg^+)$ involves brackets containing exactly j of the elements sg^- and exactly k of the elements tg^+, as in (7) and (8). In particular, $H_{j,k}(sg^-, tg^+)$ is a sum of terms of degree greater than or equal to j in s and degree greater than or equal to k in t. From (6),

$$C(sg^-, tg^+) = tg^+ + \left(\frac{\mathrm{ad}\; tg^+}{e^{\mathrm{ad}\; tg^+} - 1}\right)(sg^-) + p(s,t) \tag{20}$$

where $p(s,t)$ is a Lie series in sg^- and tg^+ whose terms have degree greater than one in sg^- and degree greater than zero in tg^+, and in particular, whose terms have degree greater than one in s and degree greater than zero in t. We set $C(sg^-, tg^+) = sh^- + th^+$ in (20). Equating the terms of degree zero in s is equivalent to the "initial conditions"

$$g_0^+ = \sum_{n\in\mathbb{N}} g_{0,n}^+ = \sum_{n\in\mathbb{N}} h_{0,n}^+ = h_0^+. \tag{21}$$

Moreover, equating the terms of degree zero in t is equivalent to the initial conditions

$$\sum_{m\in\mathbb{N}} g_{m,0}^- = \sum_{m\in\mathbb{N}} h_{m,0}^-. \tag{22}$$

Equating the terms of degree one in s in equation (20) and using the initial conditions (21) amounts to the formula

$$\begin{aligned} sh_0^- + th_1^+ &= tg_1^+ + \left(\frac{\mathrm{ad}\; th_0^+}{e^{\mathrm{ad}\; th_0^+} - 1}\right)\left(sg_0^-\right) \\ &= tg_1^+ + \left(\sum_{k\in\mathbb{N}} \frac{B_k}{k!}(\mathrm{ad}\; th_0^+)^k\right)\left(sg_0^-\right), \end{aligned}$$

where B_k, $k \in \mathbb{N}$, are the Bernoulli numbers, defined by the generating function

$$\sum_{k\in\mathbb{N}} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.$$

Equivalently (using the decomposition (11)),

$$sg_0^- = sh_0^- - s\pi^-\left(\sum_{k>0}\frac{B_k}{k!}(\mathrm{ad}\ th_0^+)^k g_0^-\right), \tag{23}$$

$$tg_1^+ = th_1^+ - s\pi^+\left(\sum_{k>0}\frac{B_k}{k!}(\mathrm{ad}\ th_0^+)^k g_0^-\right). \tag{24}$$

Equating the terms of degree one in s and one in t in equations (23) and (24) and using the initial conditions (22) is equivalent to the following information:

$$\begin{aligned} g_{0,1}^- &= h_{0,1}^- - B_1\pi^-([th_{0,0}^+, h_{0,0}^-]) \\ &= h_{0,1}^- + \frac{1}{2}t\pi^-([h_{0,0}^+, h_{0,0}^-]) \end{aligned} \tag{25}$$

and

$$\begin{aligned} g_{1,0}^+ &= h_{1,0}^+ - sB_1\pi^+([h_{0,0}^+, h_{0,0}^-]) \\ &= h_{1,0}^+ + \frac{1}{2}s\pi^+([h_{0,0}^+, h_{0,0}^-]). \end{aligned} \tag{26}$$

The conditions (21), (22), (25) and (26) are together equivalent to (15) and (16).

We will use recursion on the subscripts to construct and uniquely determine all the $g_{m,n}^{\pm}$ by equating the coefficients of appropriate powers of s and t in (18). So far, we have the following: Equating the coefficients of s^0t^n ($n \geq 1$) in (18) is equivalent to the information $g_{0,n-1}^+ = h_{0,n-1}^+$ (21); the equation for s^mt^0 ($m \geq 1$) is equivalent to $g_{m-1,0}^- = h_{m-1,0}^-$ (22); and the equation for s^1t^1 is equivalent to (25) and (26), using the special cases $g_{0,0}^+ = h_{0,0}^+$ and $g_{0,0}^- = h_{0,0}^-$ of (21) and (22). Note that (23) and (24) do not serve to uniquely construct $g_{0,n}^-$ for $n > 1$ or $g_{m,0}^+$ for $m > 1$ as desired, since we must still use the recursive procedure below to uniquely express the components of g_0^- on the right-hand sides of (23) and (24) in terms of h's. (The general recursion below will redo the cases $g_{0,1}^-$ and $g_{1,0}^+$, but we needed the precise formulas (25) and (26).)

Let $m, n > 0$. Equating the coefficients of s^mt^n in (18) is equivalent to the equation

$$sg_{m-1,n}^- + tg_{m,n-1}^+ + l_{m,n} = sh_{m-1,n}^- + th_{m,n-1}^+, \tag{27}$$

where $l_{m,n}$ is an explicit linear combination, homogeneous of degree m in s and of degree n in t, of brackets of two or more elements of the form $sg^-_{p,q}$ and $tg^+_{p,q}$ (with at least one of each of these two types) with $p < m$ and $q < n$. We equivalently have

$$sg^-_{m-1,n} = sh^-_{m-1,n} - \pi^-(l_{m,n})$$

and

$$tg^+_{m,n-1} = th^+_{m,n-1} - \pi^+(l_{m,n}).$$

Proceeding through the sequence (for example)

$$(m,n) = (1,1); (1,2), (2,1); (1,3), (2,2), (3,1); (1,4), \ldots,$$

we see that we have an effective recursive procedure for uniquely computing all the $g^\pm_{m,n}$ in terms of the $h^\pm_{m,n}$, Lie brackets, and the projections $\pi^\pm$. In particular, the map C is bijective. ∎

Problem 3.2 The map C^{-1} is given by a precise recursive procedure. We propose the following problem: Find a closed form for this map. See also Problem 3.7 below.

Remark 3.3 Theorem 3.1 and our method of proof, based on the decomposition (11), generalize to the case of a decomposition of $\mathfrak{g}$ into an arbitrary finite number of subspaces.

The proof of Theorem 3.1 yields the following more precise information about how the elements $g^\pm$ are built from the elements $h^\pm$ using commutators and the projections $\pi^\pm$, under the assumption (which we will remove in Theorem 3.6 below) that $h^\pm \in \mathfrak{g}$, i.e., that the elements $h^\pm$ do not involve s or t:

Proposition 3.4 *In the setting of Theorem 3.1, suppose that $h^\pm \in \mathfrak{g}$. Write $\mathbb{F}\langle h^\pm; \pi^\pm\rangle$ (resp., $\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle$) for the smallest Lie subalgebra of $\mathfrak{g}$ (resp., $\mathfrak{g}[[s,t]]$) containing the elements h^- and h^+ (resp., sh^- and th^+) and closed under the projections $\pi^\pm$, i.e., compatible with the decomposition (11) in the sense that it is the direct sum of its intersections with $\mathfrak{g}^\pm$ (resp., $\mathfrak{g}^\pm[[s,t]]$). We have:*

(a) The Lie algebra $\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle$ is $\mathbb{N}\times\mathbb{N}$-graded by means of the decomposition

$$\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle = \coprod_{m,n\in\mathbb{N}} \mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n} s^m t^n, \tag{28}$$

where $\mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n}$ is the subspace of $\mathbb{F}\langle h^\pm; \pi^\pm\rangle$ spanned by the elements built from commutators involving exactly m elements h^- and exactly n elements h^+, and from the projections $\pi^\pm$. (Warning: The subspaces $\mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n}$ might not be disjoint; for instance, we might have, say, $[h^+, h^-] = h^+$.)

(b) Consider the formal completion

$$\mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle = \prod_{m,n\in\mathbb{N}} \mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n} s^m t^n \tag{29}$$

of $\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle$ in $\mathfrak{g}[[s,t]]$, so that $\mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle$ is naturally a Lie subalgebra of $\mathbb{F}\langle h^\pm; \pi^\pm\rangle[[s,t]]$ stable under $\pi^\pm$. We have:

$$sg^-, tg^+ \in \mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle. \tag{30}$$

Proof: Part (a) is clear. To prove (b), it is sufficient to show that

$$(sg^-)_{m,n}, (tg^+)_{m,n} \in \mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n} s^m t^n \tag{31}$$

for all m and n (using the notation (17) for homogeneous components of elements of $\mathfrak{g}[[s,t]]$). We proceed through the proof of Theorem 3.1 and indicate the special information that we have in this situation. Formula (18) remains the same, but since the elements $h^\pm$ do not involve s or t, we do not need to consider the components $h^\pm_m$ or $h^\pm_{m,n}$. From (18) we find that (21) and (22) become, respectively:

$$\begin{aligned} tg_0^+ &= th^+, && (32)\\ s\sum_{m\in\mathbb{N}} g^-_{m,0} &= sh^-, && (33)\end{aligned}$$

so that $g_0^+ = h^+$ is independent of s and t, and $\sum g^-_{m,0} = g^-_{0,0} = h^-$ and is also independent of s and t. Also, (25) and (26) become, respectively:

$$\begin{aligned} sg^-_{0,1} &= \frac{1}{2}\pi^-([th^+, sh^-]), && (34)\\ tg^+_{1,0} &= \frac{1}{2}\pi^+([th^+, sh^-]). && (35)\end{aligned}$$

For $m, n > 0$, the right-hand side of (27) is 0, and (27) becomes:

$$sg^-_{m-1,n} + tg^+_{m,n-1} + l_{m,n} = 0, \tag{36}$$

where $l_{m,n}$ is an (explicit) linear combination as indicated in the proof. Now we just use the inductive procedure described in the proof to establish (31) by induction on (m, n). The cases $m = 0$ and $n = 0$ are covered by (32) and (33), respectively, and the case $(1, 1)$ follows from (34) and (35); $l_{1,1}$ is a multiple of $s^1t^1[h^+, h^-]$. The general inductive step is clear, and the result is proved. ∎

Remark 3.5 The proof constructs sg^- and tg^+, using the Campbell-Baker-Hausdorff series, as canonical formal series of "words" involving brackets of sh^- and th^+, and the projections $\pi^\pm$, independently of the Lie algebra $\mathfrak{g}$ or of $\pi^\pm$ or of the elements $h^\pm$. Let us call these two formal series of words the *canonical factorization series* and let us write them as

$$F^\pm(sh^-, th^+; \pi^\pm). \tag{37}$$

They are analogues, in some sense, of the Campbell-Baker-Hausdorff series.

Now we remove the assumption $h^\pm \in \mathfrak{g}$ in Proposition 3.4, using Propsition 3.4 to obtain the corresponding information in the general case. We write $\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle$ for the smallest Lie subalgebra of $\mathfrak{g}[[s,t]]$ containing sh^- and th^+ and closed under $\pi^\pm$ and we define the formal completion $\mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle$ of $\mathbb{F}\langle sh^-, th^+; \pi^\pm\rangle$ to be the vector space of formal (possibly infinite) linear combinations of "words" involving brackets of the elements sh^- and th^+, and the projections $\pi^\pm$. This space is well defined because there are only finitely many words involving s^mt^n for fixed $m, n \in \mathbb{N}$. The space $\mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle$ is clearly a Lie subalgebra of $\mathfrak{g}[[s,t]]$ stable under $\pi^\pm$. Note that this Lie algebra is an analogue of the Lie algebra $\mathbb{F}\langle\langle a, b\rangle\rangle$ in Section 2. In the special case that $h^\pm \in \mathfrak{g}$, this Lie algebra agrees with the already-defined Lie algebra (29). We have the following generalization of Proposition 3.4:

Theorem 3.6 *In the general setting of Theorem 3.1 (in the absence of the assumption $h^\pm \in \mathfrak{g}$), we have:*

$$sg^-, tg^+ \in \mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle, \tag{38}$$

and sg^- and tg^+ are given by the canonical factorization series (recall Remark 3.5):

$$sg^- = F^-(sh^-, th^+; \pi^\pm), \quad tg^+ = F^+(sh^-, th^+; \pi^\pm). \tag{39}$$

Moreover, the lowest-order terms in sg^- and tg^+ are given by:

$$sg^- = sh^- + \frac{1}{2}\pi^-([th^+, sh^-]) + u^-(s,t), \tag{40}$$

$$tg^+ = th^+ + \frac{1}{2}\pi^+([th^+, sh^-]) + u^+(s,t), \tag{41}$$

where $u^\pm(s,t) \in \mathbb{F}\langle\langle sh^-, th^+; \pi^\pm\rangle\rangle^\pm$ are formal (possibly infinite) linear combinations of words involving at least three occurrences of sh^- and th^+ (including at least one of each).

Proof: Write $\mathfrak{h}$ for the Lie algebra $\mathfrak{g}[[s,t]]$ and apply Proposition 3.4 to the Lie algebra $\mathfrak{h}$ in place of $\mathfrak{g}$ and $\mathfrak{h}[[s_1,t_1]]$ in place of $\mathfrak{g}[[s,t]]$, with s_1 and t_1 new formal variables. We find that given our elements $h^\pm \in \mathfrak{g}^\pm[[s,t]]$, we have that the formula

$$e^{s_1 g_1^-} e^{t_1 g_1^+} = e^{s_1 h^- + t_1 h^+} \tag{42}$$

determines unique elements (by Theorem 3.1)

$$g_1^\pm \in \mathfrak{h}[[s_1,t_1]], \tag{43}$$

and by Proposition 3.4, for all $m,n \in \mathbb{N}$, the coefficient of $s_1^m t_1^n$ in $s_1 g_1^-$ and in $t_1 g_1^+$ lies in $\mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n}$, where $\mathbb{F}\langle h^\pm; \pi^\pm\rangle$ and $\mathbb{F}\langle h^\pm; \pi^\pm\rangle_{m,n}$ are defined as in Proposition 3.4 in the present case. Moreover, $s_1 g_1^-$ and $t_1 g_1^+$ are expressed by the canonical factorization series in terms of $s_1 h^-$ and $t_1 h^+$, and $\pi^\pm$, and the low-order terms with respect to s_1 and t_1 in $g_1^\pm$ are given by (15) and (16) with s_1 and t_1 in place of s and t and with $\mathbb{F}\langle h^\pm; \pi^\pm\rangle$ in place of $\mathfrak{g}$. We may set $s = s_1$ and $t = t_1$ in (42), and we see that the elements $g^\pm$ are determined from the elements

$$g_1^\pm \in \mathbb{F}\langle h^\pm; \pi^\pm\rangle[[s_1,t_1]] \tag{44}$$

(which are uniquely determined by the formula (42)), by the specialization

$$g^\pm = g_1^\pm|_{s_1=s,\, t_1=t}. \tag{45}$$

Moreover,

$$s_1 g_1^- = s_1 h^- + \frac{1}{2} s_1 t_1 \pi^-([h^+, h^-]) + s_1 t_1 r_1^-(s_1,t_1), \tag{46}$$

$$t_1 g_1^+ = t_1 h^+ + \frac{1}{2} s_1 t_1 \pi^+([h^+, h^-]) + s_1 t_1 r_1^+(s_1,t_1), \tag{47}$$

where

$$r_1^{\pm}(s_1,t_1) \in \mathbb{F}\langle h^{\pm};\pi^{\pm}\rangle^{\pm}[[s_1,t_1]], \quad r_1^{\pm}(0,0)=0, \tag{48}$$

and for all $m,n \geq 1$ with either m or $n \geq 2$, the coefficient of $s_1^m t_1^n$ in $s_1 t_1 r_1^{\pm}(s_1,t_1)$ is a linear combination of elements of $\mathbb{F}\langle h^{\pm};\pi^{\pm}\rangle$ built according to the canonical factorization series from commutators involving exactly m elements h^- and exactly n elements h^+, and from the projections $\pi^{\pm}$. This gives the desired result. ∎

Problem 3.7 Find closed forms for the canonical factorization series $F^{\pm}$. (Cf. Problem 3.2 and the classical formula (6) for the Campbell-Baker-Hausdorff series.)

Remark 3.8 Here we give an alternate, direct, simple proof of the uniqueness of the factors in the product $e^{sg^-}e^{tg^+}$ in (13) (i.e., the injectivity of the map C in Theorem 3.1), under the extra hypothesis that the subspaces $\mathfrak{g}^{\pm}$ are Lie subalgebras (but see the next remark for the removal of this extra hypothesis). The following argument also works more generally for the analogous uniqueness when the Lie algebra $\mathfrak{g}$ is given as a finite direct sum of any number of Lie subalgebras. We use the Poincaré-Birkhoff-Witt theorem. Write

$$P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}^-)$$

for the projection with respect to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{g}^-) \oplus U(\mathfrak{g}^-)U(\mathfrak{g}^+)\mathfrak{g}^+ = U(\mathfrak{g}^-) \oplus U(\mathfrak{g})\mathfrak{g}^+,$$

coming from the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{g}^-) \otimes U(\mathfrak{g}^+),$$

and extend P canonically to $U(\mathfrak{g})[[s,t]]$. Given

$$e^{sg^-}e^{tg^+} = e^{sg_1^-}e^{tg_1^+} \tag{49}$$

($g_1^{\pm} \in \mathfrak{g}^{\pm}[[s,t]]$), simply apply P to get $e^{sg^-} = e^{sg_1^-}$ and hence $g^- = g_1^-$, and from this, $g^+ = g_1^+$.

Remark 3.9 Here we remove the extra hypothesis in Remark 3.8 that the subspaces $\mathfrak{g}^{\pm}$ be Lie subalgebras, proving the uniqueness in Theorem 3.1 in general; as in Remark 3.8, this argument works in the general setting of Remark 3.3. Let

$$\lambda : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$

be the standard symmetrization map—a linear isomorphism by the Poincaré-Birkhoff-Witt theorem—determined by:

$$\lambda(g_1 \cdots g_k) = \frac{1}{k!} \sum_{\sigma} g_{\sigma(1)} \cdots g_{\sigma(k)},$$

where $k \geq 0$, $g_1, \ldots, g_k \in \mathfrak{g}$ and σ ranges through the symmetric group on k letters. Then by the Poincaré-Birkhoff-Witt theorem,

$$\begin{aligned} U(\mathfrak{g}) &= \lambda(S(\mathfrak{g}^-)) \otimes \lambda(S(\mathfrak{g}^+)) \\ &= \lambda(S(\mathfrak{g}^-)) \oplus \lambda(S(\mathfrak{g}^-))\lambda(\mathfrak{g}^+ S(\mathfrak{g}^+)). \end{aligned}$$

Let

$$P : U(\mathfrak{g}) \longrightarrow \lambda(S(\mathfrak{g}^-))$$

be the corresponding projection, and extend P canonically to the projection

$$P : U(\mathfrak{g})[[s,t]] \longrightarrow \lambda(S(\mathfrak{g}^-))[[s,t]].$$

Now with $g^- \in \mathfrak{g}^-[[s,t]]$ as above, we have

$$e^{sg^-} \in \lambda(S(\mathfrak{g}^-))[[s,t]]$$

(and similarly for tg^+). Indeed, the coefficient of each monomial in s and t in e^{sg^-} coincides with the coefficient of the same monomial in a suitable *finite* linear combination of powers of sg^-, and for any $k \geq 0$,

$$(sg^-)^k \in \lambda(S(\mathfrak{g}^-))[[s,t]],$$

since the map λ extends canonically to the natural map

$$\lambda : S(\mathfrak{g})[[s,t]] \longrightarrow U(\mathfrak{g})[[s,t]],$$

and $(sg^-)^k$ is the image of $(sg^-)^k$ viewed as an element of $S(\mathfrak{g}^-)[[s,t]]$. Given (49), we can now apply P just as in Remark 3.8 to get $e^{sg^-} = e^{sg_1^-}$, giving the uniqueness.

Remark 3.10 Recall that the nontrivial part of the Campbell-Baker-Hausdorff theorem is that the element $C(a, b)$ (see formula (5)) of the universal enveloping algebra of the free Lie algebra over a and b is in fact a Lie series. In the proof of bijectivity in Theorem 3.1, in general, the operations take place in $U(\mathfrak{g})[[s, t]]$; however, the projections $\pi^{\pm}$ are not defined on $U(\mathfrak{g})[[s, t]]$. Rather, it is necessary to use the fact that $C(sg^-, tg^+)$ is a Lie series, to take brackets in the Lie algebra $\mathfrak{g}[[s, t]]$, and then to project using $\pi^{\pm}$. One cannot use (associative) words in $g^{\pm}$ to obtain $C^{-1}(sh^- + th^+)$, in contrast with the situation for $C(sg^-, tg^+)$. However, Theorem 3.6 shows that $C^{-1}(sh^- + th^+)$ can be obtained using words in sh^- and th^+ and the canonical projections $\pi^{\pm}$, i.e., the correct setting for $C^{-1}(sh^- + th^+)$ is $\mathbb{F}\langle\langle sh^-, th^+; \pi^{\pm}\rangle\rangle$, as opposed to $U(\mathfrak{g})[[s, t]]$ for $C(sg^-, tg^+)$.

Remark 3.11 The method used to prove that the map C in Theorem 3.1 is bijective is similar to the method used to prove the "formal uniformization" result Theorem 2.2.4 in [H3] for a certain representation of the Virasoro algebra, given by $L_n \mapsto -x^{n+1}\frac{d}{dx} \in \mathrm{End}(\mathbb{C}[x, x^{-1}])$ and $c = 0$, and to prove the analogous result, Theorem 2.3.4, in [B1], for an analogous representation of the $N = 1$ Neveu-Schwarz algebra. The method is similar in that once one has an appropriate equation involving formal series and one has appropriate projections, one can solve the equation recursively. However, the settings in the proofs for these two cases are very different from the setting in the present proof: In [H3] and [B1] the proofs take place in certain formal function algebras, whereas in the present paper the proof takes place in a certain Lie algebra. Note the subtle issue that in a universal enveloping algebra or formal extension there are no appropriate projections available for the use of this method (recall Remark 3.10). In fact a crucial observation in this paper is that even though there is no such setting involving universal enveloping algebras, one can in fact find a very general setting to which the method can be applied, a setting very different from the ones in [H3] and [B1]. As a benefit, we are able to obtain the factorization results in [H3] and [B1] in a uniform, simple way, without the need to pass to a central extension (see Applications 4.3 and 4.7 below, Theorem 2.2.4, Proposition 4.2.1 and Corollary 4.2.2 in [H3], and Theorem 2.3.4, Proposition 2.6.1 and Corollary 2.6.2 in [B1]).

Corollary 3.12 (Formal Algebraic Uniformization) *There exists a unique bijection*

$$\Psi : t\mathfrak{g}^+[[s,t]] \times s\mathfrak{g}^-[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]] \tag{50}$$

such that for $g^\pm \in \mathfrak{g}^\pm[[s,t]]$,

$$e^{tg^+} e^{sg^-} = e^{s\Psi^-} e^{t\Psi^+} \tag{51}$$

with $s\Psi^- = \pi^- \circ \Psi(tg^+, sg^-)$ *and* $t\Psi^+ = \pi^+ \circ \Psi(tg^+, sg^-)$. *Moreover, the lowest order terms of the* $\Psi^\pm$ *are described as follows:*

$$\Psi^- = g^- + t\pi^-([g^+, g^-]) + t\mathcal{P}^-(s,t) \tag{52}$$
$$\Psi^+ = g^+ + s\pi^+([g^+, g^-]) + s\mathcal{P}^+(s,t) \tag{53}$$

where $\mathcal{P}^\pm(s,t) \in \mathfrak{g}^\pm[[s,t]]$ *and* $\mathcal{P}^\pm(0,0) = 0$.

Proof: Let C' be the analogue of the map C (see (12), (13)) with the roles of s and t and of $\mathfrak{g}^-$ and $\mathfrak{g}^+$ reversed, and let σ be the isomorphism given by

$$\begin{array}{rcl} \sigma : t\mathfrak{g}^+[[s,t]] \oplus s\mathfrak{g}^-[[s,t]] & \longrightarrow & s\mathfrak{g}^-[[s,t]] \oplus t\mathfrak{g}^+[[s,t]] \\ (th^+, sh^-) & \mapsto & (sh^-, th^+). \end{array}$$

Then

$$\Psi = C^{-1} \circ \sigma \circ C' : t\mathfrak{g}^+[[s,t]] \times s\mathfrak{g}^-[[s,t]] \longrightarrow s\mathfrak{g}^-[[s,t]] \times t\mathfrak{g}^+[[s,t]]$$

is a bijection satisfying (51). The conditions given by equations (52) and (53) for the lowest order terms follow from equations (14), (15), and (16). ∎

Theorem 3.6 along with the Campbell-Baker-Hausdorff theorem of course gives the corresponding additional information about the elements $\Psi^\pm$ in Corollary 3.12:

Corollary 3.13 *We have:*

$$s\Psi^-, t\Psi^+ \in \mathbb{F}\langle\langle sg^-, tg^+; \pi^\pm \rangle\rangle \tag{54}$$

and $s\Psi^-$ *and* $t\Psi^+$ *are given by canonical formal series, which we write as:*

$$s\Psi^- = G^-(sg^-, tg^+; \pi^\pm), \quad t\Psi^+ = G^+(sg^-, tg^+; \pi^\pm). \tag{55}$$

The lowest-order terms are given by:

$$G^-(sg^-, tg^+; \pi^\pm) = sg^- + \pi^-([tg^+, sg^-]) + v^-(s,t), \tag{56}$$

$$G^+(sg^-, tg^+; \pi^\pm) = tg^+ + \pi^+([tg^+, sg^-]) + v^+(s,t), \tag{57}$$

where $v^\pm(s,t) \in \mathbb{F}\langle\langle sg^-, tg^+; \pi^\pm\rangle\rangle^\pm$ are formal (possibly infinite) linear combinations of words involving at least three occurrences of sg^- and tg^+ (including at least one of each). ∎

Problem 3.14 Let us call the two series $G^\pm$ the *formal algebraic uniformization series*. They are essentially compositions, in a suitable sense, of the Campbell-Baker-Hausdorff series and the canonical factorization series, incorporating the twist σ in the proof of Corollary 3.12. Find closed forms for the formal algebraic uniformization series. (Cf. Problem 3.7.)

4 Applications to $\mathbb{Z}$-graded Lie algebras

In this section we apply our formal algebraic uniformization result, Corollary 3.12, to the Lie algebra $\mathfrak{g}$ consisting of certain formal infinite series with coefficients in a $\mathbb{Z}$-graded Lie algebra $\mathfrak{p}$. We then give applications for $\mathfrak{p}$ an affine Lie algebra, the Virasoro algebra and a Grassmann envelope of the $N = 1$ Neveu-Schwarz algebra. For the latter two applications, we discuss the importance of these results to conformal and superconformal field theory, and we point out that the result also applies to Grassmann envelopes of the $N > 1$ Neveu-Schwarz algebras.

We continue to work over our field $\mathbb{F}$ of characteristic zero. Let $\mathfrak{p} = \coprod_{j\in\mathbb{Z}} \mathfrak{p}_j$ be a $\mathbb{Z}$-graded Lie algebra. (Note that if, for example, $\mathfrak{p}$ is given as $\frac{1}{T}\mathbb{Z}$-graded for some positive integer T, then by regrading, we can always consider $\mathfrak{p}$ as $\mathbb{Z}$-graded.) We consider the following vector space decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \left(\coprod_{j<0} \mathfrak{p}_j\right) \oplus \mathfrak{p}_0 \oplus \left(\coprod_{j>0} \mathfrak{p}_j\right) = \mathfrak{p}^- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}^+.$$

Let $\mathcal{A}_j$ (resp., $\mathcal{B}_j$) be commuting formal variables for $j \in \mathbb{Z}$, and $j > 0$ (resp., $j < 0$). Consider the corresponding polynomial algebra $\mathbb{F}[\mathcal{A}, \mathcal{B}]$. We define the *order* of each $\mathcal{A}_j$ and each $\mathcal{B}_j$ to be one. This induces three gradings

by nonnegative integers, called the *order in the $\mathcal{A}_j$'s*, the *order in the $\mathcal{B}_j$'s* and the *total order in the $\mathcal{A}_j$'s and $\mathcal{B}_j$'s*, defined in the obvious ways.

Now consider the space $\mathfrak{p}[\mathcal{A},\mathcal{B}]$ of polynomials in the $\mathcal{A}_j$'s and $\mathcal{B}_j$'s with coefficients in $\mathfrak{p}$, equipped with the three gradings by order. Also consider the corresponding space $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$ of formal power series. Make $\mathfrak{p}[\mathcal{A},\mathcal{B}]$ and $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$ into Lie algebras in the canonical ways, and note that $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$ has a vector space decomposition

$$\mathfrak{p}[[\mathcal{A},\mathcal{B}]] = \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]] \tag{58}$$

into three Lie subalgebras $\mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$, $\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]]$ and $\mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$.

We are now ready to apply our results from the preceding section. We want to exponentiate suitable elements of the form $(p_1, p_2, ...) \in \prod_{j>0} \mathfrak{p}_j$, with possibly infinitely many p_j's nonzero, and corresponding elements of $\prod_{j<0} \mathfrak{p}_j$. But products of such exponentials are not well defined even if we use formal variables such as s and t as in the previous section. However, if we use an infinite number of formal variables, one for each homogeneous subspace, then we can multiply the exponentials.

Therefore, fix

$$(p_1, p_2, ...) \in \prod_{j>0} \mathfrak{p}_j \qquad \text{and} \qquad (p_{-1}, p_{-2}, ...) \in \prod_{j<0} \mathfrak{p}_j, \tag{59}$$

and define corresponding elements

$$g^+ = \sum_{j>0} \mathcal{A}_j p_j \qquad \text{and} \qquad g^- = \sum_{j<0} \mathcal{B}_j p_j \tag{60}$$

of $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$. We will consider exponentials of these elements.

We will apply Corollary 3.12 and Corollary 3.13 to the Lie algebra $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$ with the decomposition $\mathfrak{p}[[\mathcal{A},\mathcal{B}]] = \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]] \oplus (\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]])$ with corresponding projectins $\pi^- : \mathfrak{p}[[\mathcal{A},\mathcal{B}]] \to \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$, and $\pi^{0,+} : \mathfrak{p}[[\mathcal{A},\mathcal{B}]] \to (\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]])$, and then we will apply Theorem 3.6 to the Lie algebra $\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$ with the indicated decomposition and corresponding projections $\pi^0 : \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]] \to \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]]$ and $\pi^+ : \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]] \to \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$.

Corollary 4.1 *With the notation as above, there exist unique elements* $\Psi^- \in \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$*,* $\Psi^0 \in \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]]$ *and* $\Psi^+ \in \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$ *of the form*

$$\begin{aligned} \Psi^- &= \sum_{j<0} \mathcal{B}_j p_j + \mathcal{Q}^-(\mathcal{A},\mathcal{B}), && (61)\\ \Psi^0 &= \mathcal{Q}^0(\mathcal{A},\mathcal{B}), && (62)\\ \Psi^+ &= \sum_{j>0} \mathcal{A}_j p_j + \mathcal{Q}^+(\mathcal{A},\mathcal{B}), && (63) \end{aligned}$$

where $\mathcal{Q}^-(\mathcal{A},\mathcal{B}) \in \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$*,* $\mathcal{Q}^0(\mathcal{A},\mathcal{B}) \in \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]]$ *and* $\mathcal{Q}^+(\mathcal{A},\mathcal{B}) \in \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$*, and these elements contain only terms of order at least one in the* $\mathcal{A}_j$*'s and order at least one in the* $\mathcal{B}_j$*'s, such that*

$$e^{g^+} e^{g^-} = e^{\Psi^-} e^{\Psi^+} e^{\Psi^0} \tag{64}$$

in $U(\mathfrak{p})[[\mathcal{A},\mathcal{B}]]$*. (Note that the right-hand side of (64) is well defined for any elements of the form (61)–(63).) Moreover,*

$$\Psi^-, \Psi^0, \Psi^+ \in \mathbb{F}\langle\langle g^-, g^+; \pi^-, \pi^0, \pi^+\rangle\rangle \tag{65}$$

(using obvious notation), and we have

$$\begin{aligned} \Psi^- &= \sum_{j<0} \mathcal{B}_j p_j + \sum_{\substack{j>0,\, m<0\\ j+m<0}} \mathcal{A}_j \mathcal{B}_m \,[p_j, p_m] + \mathcal{P}^-(\mathcal{A},\mathcal{B}), && (66)\\ \Psi^0 &= \sum_{j>0} \mathcal{A}_j \mathcal{B}_{-j} \,[p_j, p_{-j}] + \mathcal{P}^0(\mathcal{A},\mathcal{B}), && (67)\\ \Psi^+ &= \sum_{j>0} \mathcal{A}_j p_j + \sum_{\substack{j>0,\, m<0\\ j+m>0}} \mathcal{A}_j \mathcal{B}_m \,[p_j, p_m] + \mathcal{P}^+(\mathcal{A},\mathcal{B}), && (68) \end{aligned}$$

where $\mathcal{P}^-(\mathcal{A},\mathcal{B})$*,* $\mathcal{P}^0(\mathcal{A},\mathcal{B})$*,* $\mathcal{P}^+(\mathcal{A},\mathcal{B}) \in \mathfrak{p}[[\mathcal{A},\mathcal{B}]]$ *each contain only terms of total order three or more in the* $\mathcal{A}_j$*'s and* $\mathcal{B}_j$*'s, with order at least one in the* $\mathcal{A}_j$*'s and at least one in the* $\mathcal{B}_j$*'s.*

Proof: The uniqueness of Ψ^-, Ψ^0 and Ψ^+ is immediate from the argument in Remark 3.8, applied first to the decomposition

$$U(\mathfrak{p}) = U(\mathfrak{p}^-) \otimes U(\mathfrak{p}^+) \otimes U(\mathfrak{p}_0)$$

and the corresponding projection

$$U(\mathfrak{p})[[\mathcal{A},\mathcal{B}]] \longrightarrow U(\mathfrak{p}^-)U(\mathfrak{p}^+)[[\mathcal{A},\mathcal{B}]].$$

This gives the uniqueness of the product $e^{\Psi^-}e^{\Psi^+}$, and the analogous consideration of the decomposition

$$U(\mathfrak{p}^-)U(\mathfrak{p}^+) = U(\mathfrak{p}^-) \oplus U(\mathfrak{p}^-)U(\mathfrak{p}^+)\mathfrak{p}^+$$

inside $U(\mathfrak{p})$ then completes the uniqueness.

Note that $g^- \in \mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$ and $g^+ \in \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]] \subset \mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$. By Corollaries 3.12 and 3.13, there exist unique elements $\tilde{\Psi}^- \in (\mathfrak{p}^-[[\mathcal{A},\mathcal{B}]])[[s,t]]$ and $\tilde{\Psi}^{0,+} \in (\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]])[[s,t]]$ such that

$$e^{tg^+}e^{sg^-} = e^{s\tilde{\Psi}^-}e^{t\tilde{\Psi}^{0,+}},$$

and $s\tilde{\Psi}^-, t\tilde{\Psi}^{0,+} \in \mathbb{F}\langle\langle sg^-, tg^+; \pi^-, \pi^{0,+}\rangle\rangle \subseteq \mathbb{F}\langle\langle sg^-, tg^+; \pi^-, \pi^0, \pi^+\rangle\rangle$, and these elements satisfy (56) and (57), respectively, applied to this case. For any element of $\mathbb{F}\langle\langle sg^-, tg^+; \pi^-, \pi^{0,+}\rangle\rangle$, we can substitute 1 for both s and t and the result is an element of $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$. Thus $\Psi^- = s\tilde{\Psi}^- \mid_{s=t=1}$ and $\Psi^{0,+} = t\tilde{\Psi}^{0,+} \mid_{s=t=1}$ are well-defined elements of $\mathfrak{p}^-[[\mathcal{A},\mathcal{B}]]$ and $\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]] \oplus \mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$, respectively, and e^{Ψ^-} and $e^{\Psi^{0,+}}$ are well defined in $U(\mathfrak{p})[[\mathcal{A},\mathcal{B}]]$. Moreover, Ψ^- and $\Psi^{0,+}$ satisfy the analogues of (56) and (57), respectively, applied to this case, and thus Ψ^- satisfies (66).

Now let $h^+ = \pi^+(\Psi^{0,+})$ and $h^0 = \pi^0(\Psi^{0,+})$. By Theorems 3.1 and 3.6, there exist unique $\tilde{\Psi}^+ \in (\mathfrak{p}^+[[\mathcal{A},\mathcal{B}]])[[s,t]]$ and $\tilde{\Psi}^0 \in (\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]])[[s,t]]$ such that

$$e^{s\tilde{\Psi}^+}e^{t\tilde{\Psi}^0} = e^{sh^+ + th^0},$$

and $s\tilde{\Psi}^+, t\tilde{\Psi}^0 \in \mathbb{F}\langle\langle sh^+, th^0; \pi^+, \pi^0\rangle\rangle$ and $s\tilde{\Psi}^+$ and $t\tilde{\Psi}^0$ satisfy (40) and (41), respectively, applied to this case. Again we can substitute 1 for both s and t and the result is an element of $\mathfrak{p}[[\mathcal{A},\mathcal{B}]]$. Thus we have that $\Psi^+ = s\tilde{\Psi}^+ \mid_{s=t=1}$ and $\Psi^0 = t\tilde{\Psi}^0 \mid_{s=t=1}$ are well defined elements of $\mathfrak{p}^+[[\mathcal{A},\mathcal{B}]]$ and $\mathfrak{p}_0[[\mathcal{A},\mathcal{B}]]$, respectively, and e^{Ψ^-} and $e^{\Psi^{0,+}}$ are well defined in $U(\mathfrak{p})[[\mathcal{A},\mathcal{B}]]$. Moreover, $\Psi^+, \Psi^0 \in \mathbb{F}\langle\langle h^+, h^0; \pi^+, \pi^0\rangle\rangle \subseteq \mathbb{F}\langle\langle g^-, g^+; \pi^-, \pi^+, \pi^0\rangle\rangle$, and by (40) and (41) applied to this case, Ψ^0 and Ψ^+ satisfy (67) and (68), respectively. ∎

Now we specialize to the following situations:

Application 4.2 *Affine Lie algebras.* Let $\mathfrak{l}$ be a finite dimensional Lie algebra equipped with an $\mathfrak{l}$-invariant symmetric bilinear form $(\cdot,\cdot)$ and consider

the corresponding affine Lie algebra $\hat{\mathfrak{l}} = \mathfrak{l} \otimes \mathbb{C}[x, x^{-1}] \oplus \mathbb{C}c$, with commutation relations

$$[g \otimes x^m, h \otimes x^n] = [g, h] \otimes x^{m+n} + (g, h) m \delta_{m+n,0} \mathbf{k},$$

and $\mathbf{k}$ central. Consider also the natural $\mathbb{Z}$-grading $\hat{\mathfrak{l}} = \coprod_{n \in \mathbb{Z}} \hat{\mathfrak{l}}_n$.

Fix $h_j \in \mathfrak{l}$ for $j \in \mathbb{Z} \setminus \{0\}$ and let $\mathcal{A}_j$ for $j > 0$ and $\mathcal{B}_j$ for $j < 0$ be commuting formal variables. We can now apply Corollary 4.1 with $\mathfrak{p} = \hat{\mathfrak{l}}$, $\mathfrak{p}_j = \hat{\mathfrak{l}}_j$ for $j \in \mathbb{Z}$ and $p_j = h_j \otimes x^j$ for $j \in \mathbb{Z} \setminus \{0\}$. Thus

$$g^+ = \sum_{j>0} \mathcal{A}_j \; h_j \otimes x^j \qquad \text{and} \qquad g^- = \sum_{j<0} \mathcal{B}_j \; h_j \otimes x^j,$$

and we have that there exist unique $\Psi^- \in \hat{\mathfrak{l}}^-[[\mathcal{A}, \mathcal{B}]]$, $\Psi^0 \in \hat{\mathfrak{l}}_0[[\mathcal{A}, \mathcal{B}]]$, and $\Psi^+ \in \hat{\mathfrak{l}}^+[[\mathcal{A}, \mathcal{B}]]$ satisfying (61) - (63) such that

$$\exp\left(\sum_{j>0} \mathcal{A}_j \; h_j \otimes x^j\right) \exp\left(\sum_{j<0} \mathcal{B}_j \; h_j \otimes x^j\right) = e^{\Psi^-} e^{\Psi^+} e^{\Psi^0}. \tag{69}$$

Also, formula (65) holds. Furthermore, we can write $\Psi^0 \in \hat{\mathfrak{l}}_0[[\mathcal{A}, \mathcal{B}]]$ as $\Psi^0 = \Psi^0_{\mathfrak{l}} + \Psi^0_{\mathbf{k}}$ where $\Psi^0_{\mathfrak{l}} \in \mathfrak{l}[[\mathcal{A}, \mathcal{B}]]$ and $\Psi^0_{\mathbf{k}} \in (\mathbb{C}\mathbf{k})[[\mathcal{A}, \mathcal{B}]]$. But since $\mathbf{k}$ is central, we can write the last exponential in equation (69) as

$$e^{\Psi^0} = \exp\left(\Psi^0_{\mathfrak{l}} + \Psi^0_{\mathbf{k}}\right) = \exp\left(\Psi^0_{\mathfrak{l}}\right) \exp\left(\Psi^0_{\mathbf{k}}\right).$$

In addition, we have

$$\begin{aligned}
\Psi^- &= \sum_{j<0} \mathcal{B}_j \; h_j \otimes x^j + \\
&\qquad \sum_{\substack{j>0,\, m<0 \\ j+m<0}} \mathcal{A}_j \mathcal{B}_m \, [h_j, h_m] \otimes x^{j+m} + \mathcal{P}^-(\mathcal{A}, \mathcal{B}), && (70) \\
\Psi^+ &= \sum_{j>0} \mathcal{A}_j \; h_j \otimes x^j + \\
&\qquad \sum_{\substack{j>0,\, m<0 \\ j+m>0}} \mathcal{A}_j \mathcal{B}_m \, [h_j, h_m] \otimes x^{j+m} + \mathcal{P}^+(\mathcal{A}, \mathcal{B}), && (71) \\
\Psi^0_{\mathfrak{l}} &= \sum_{j>0} \mathcal{A}_j \mathcal{B}_{-j} \, [h_j, h_{-j}] + \mathcal{P}^0_{\mathfrak{l}}(\mathcal{A}, \mathcal{B}), && (72) \\
\Psi^0_{\mathbf{k}} &= \sum_{j>0} \mathcal{A}_j \mathcal{B}_{-j} \, (h_j, h_{-j}) \, j \mathbf{k} + \mathcal{P}^0_{\mathbf{k}}(\mathcal{A}, \mathcal{B}) && (73)
\end{aligned}$$

where $\mathcal{P}^-(\mathcal{A},\mathcal{B})$, $\mathcal{P}^+(\mathcal{A},\mathcal{B})$, $\mathcal{P}^0_{\mathfrak{l}}(\mathcal{A},\mathcal{B})$, $\mathcal{P}^0_{\mathbf{k}}(\mathcal{A},\mathcal{B}) \in \hat{\mathfrak{l}}[[\mathcal{A},\mathcal{B}]]$ each contain only terms of total order three or more in the $\mathcal{A}_j$'s and $\mathcal{B}_j$'s, with order at least one in the $\mathcal{A}_j$'s and at least one in the $\mathcal{B}_j$'s.

Application 4.3 *The Virasoro algebra.* Take $\mathfrak{p} = \mathfrak{v}$ to be the Virasoro algebra. With the usual basis, L_m for $m \in \mathbb{Z}$ and c, the commutation relations for $\mathfrak{v}$ are

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c, \\ [L_m, c] &= 0 \end{aligned}$$

for $m, n \in \mathbb{Z}$. Consider the natural $\mathbb{Z}$-grading $\mathfrak{v} = \coprod_{j\in\mathbb{Z}} \mathfrak{v}_j$, where $\mathfrak{v}_j = \mathbb{C}L_j$ for $j \in \mathbb{Z} \setminus \{0\}$ and $\mathfrak{v}_0 = \mathbb{C}L_0 \oplus \mathbb{C}c$.

Now take $p_j = L_j$ for $j \in \mathbb{Z} \smallsetminus \{0\}$, and as usual let $\mathcal{A}_j$ and $\mathcal{B}_{-j}$ be commuting formal variables for $j \in \mathbb{Z}$, $j > 0$. (Note that we could take $p_j = c_j L_j$ where c_j are complex variables. However, in expressions such as $\mathcal{A}_j c_j L_j$, we can always absorb the complex variables c_j into the formal variables $\mathcal{A}_j$. In fact, in applications such as in [H1] - [H3], under suitable conditions, one eventually wants to specialize the formal variables to be complex numbers. Thus we have not sacrificed any generality by setting $c_j = 1$.)

Now take

$$g^+ = \sum_{j>0} \mathcal{A}_j L_j \qquad \text{and} \qquad g^- = \sum_{j<0} \mathcal{B}_j L_j.$$

Applying Corollary 4.1, we see that there exist unique $\Psi^- \in \mathfrak{v}^-[[\mathcal{A},\mathcal{B}]]$, $\Psi^0 \in \mathfrak{v}_0[[\mathcal{A},\mathcal{B}]]$ and $\Psi^+ \in \mathfrak{v}^+[[\mathcal{A},\mathcal{B}]]$ satisfying (61) - (63), such that

$$\exp\left(\sum_{j>0} \mathcal{A}_j L_j\right) \exp\left(\sum_{j<0} \mathcal{B}_j L_j\right) = e^{\Psi^-} e^{\Psi^+} e^{\Psi^0}, \tag{74}$$

and formula (65) holds. Let us write

$$\Psi^- = \sum_{j<0} \Psi_j L_j, \qquad \Psi^+ = \sum_{j>0} \Psi_j L_j$$

and

$$\Psi^0 = \Psi_0 L_0 + \Gamma c$$

where $\Psi_j, \Gamma \in \mathbb{C}[[\mathcal{A}, \mathcal{B}]]$ for $j \in \mathbb{Z}$. Since c is central, (74) is equal to

$$\exp\left(\sum_{j<0} \Psi_j L_j\right) \exp\left(\sum_{j>0} \Psi_j L_j\right) e^{\Psi_0 L_0} e^{\Gamma c},$$

and for $j > 0$, we have

$$\Psi_{-j} = \mathcal{B}_{-j} + \sum_{m>j} \mathcal{A}_{-j+m} \mathcal{B}_{-m}(-j+2m) + \mathcal{P}_{-j}(\mathcal{A}, \mathcal{B}), \tag{75}$$

$$\Psi_j = \mathcal{A}_j + \sum_{m>0} \mathcal{A}_{j+m} \mathcal{B}_{-m}(j+2m) + \mathcal{P}_j(\mathcal{A}, \mathcal{B}), \tag{76}$$

$$\Psi_0 = \sum_{m>0} 2\mathcal{A}_m \mathcal{B}_{-m} m + \mathcal{P}_0(\mathcal{A}, \mathcal{B}), \tag{77}$$

$$\Gamma = \sum_{m>0} \mathcal{A}_m \mathcal{B}_{-m} \frac{(m^3 - m)}{12} + \Gamma_0(\mathcal{A}, \mathcal{B}), \tag{78}$$

where $\mathcal{P}_j(\mathcal{A}, \mathcal{B}), \Gamma_0(\mathcal{A}, \mathcal{B}) \in \mathbb{C}[[\mathcal{A}, \mathcal{B}]]$, for $j \in \mathbb{Z}$, contain only terms of total order three or more in the $\mathcal{A}_m$'s and $\mathcal{B}_m$'s with order at least one in the $\mathcal{A}_m$'s and at least one in the $\mathcal{B}_m$'s.

Remark 4.4 In conformal field theory, equation (74) corresponds to calculating the uniformizing function to obtain a canonical sphere with tubes from the sewing together of two canonical spheres with tubes in the moduli space of spheres with tubes under global conformal equivalence. This moduli space along with the sewing operation is the geometric structure underlying a geometric vertex operator algebra [H1] - [H3]. Equation (74) also corresponds to a certain change of variables and "normal ordering" of the operators L_j generated by the Virasoro element in an (algebraic) vertex operator algebra, where by "normal ordering" we mean ordering the operators L_j so as to first act by the operators L_j for $j > 0$ and then act by the operators L_j for $j < 0$. The correspondence between these two procedures, one geometric and the other algebraic, is necessary for the proof of the isomorphism between the category of vertex operator algebras and the category of geometric vertex operator algebras [H2].

Remark 4.5 The results about the formal series $\Psi_j, \Gamma \in \mathbb{C}[[\mathcal{A}, \mathcal{B}]]$ for $j \in \mathbb{Z}$, given in equations (75) - (78) above—the explicit results about the lowest

order terms and the qualitative information about the higher order terms—are exactly the results necessary for the proof of the isomorphism between the category of geometric vertex operator algebras and the category of algebraic vertex operator algebras. Equivalent results were proved by Huang, first in [H1] and then in Theorem 2.2.4[4], Proposition 2.2.5[5], Proposition 4.2.1 and Corollary 4.2.2 of [H3]. However, the quantitative information in equations (75) - (78) is much more explicit than the equivalent information given in [H3]. The main difference between equations (75) - (78) and the analogous results given in [H3] is that in (75) - (78), the terms of total order two in the $\mathcal{A}_j$'s and $\mathcal{B}_j$'s are given explicitly while in [H3] this information for the Ψ_j's and Γ is presented in (the corrected forms of) equations (2.2.11) and (2.2.12) (see footnote 4) using the representation of the Virasoro algebra given by $L_n = -x^{n+1}\frac{d}{dx} \in \mathrm{End}(\mathbb{C}[x,x^{-1}])$ and $c = 0$. In order to recover equations (75) - (78) above from the results in [H3], one must perform several operations, pick out coefficients, and then use Proposition 4.2.1 and Corollary 4.2.2 in [H3], allowing one to lift the results from the particular representation to the algebra. For example, a shorter and more straightforward proof of Proposition 3.5.2 in [H3] than that originally given can be obtained using equations (75) - (78) above. This proposition states that the meromorphic tangent space of the moduli space of spheres with one incoming tube and one outgoing tube carries the structure of a Virasoro algebra with central charge zero.

We now show how Corollary 4.1 can be applied to Lie superalgebras. We will use the notion of a Grassmann envelope of a Lie superalgebra. These are Lie algebras to which we will apply Corollary 4.1.

We will be interested in $\frac{1}{2}\mathbb{Z}$-graded vector spaces also equipped with a compatible $\mathbb{Z}_2$-grading. To distinguish, we will denote the $\mathbb{Z}_2$-grading using superscripts. For a $\mathbb{Z}_2$-graded vector space $V = V^0 \oplus V^1$, define the *sign function* η on the homogeneous subspaces of V by $\eta(v) = i$ for $v \in V^i$, $i = 0, 1$. If $\eta(v) = 0$, we say that v is *even*, and if $\eta(v) = 1$, we say that v is *odd*.

[4]We take this opportunity to correct a misprint in the formulas (2.2.11) and (2.2.12) of Theorem 2.2.4 in [H3]. The first two terms in the right-hand side of (2.2.11) should be replaced by $\alpha_0^{-1}(f_{\mathcal{B}}^{(2)})^{-1}(\frac{1}{\alpha_0 x})$ and the first two terms in the right-hand side of (2.2.12) should be replaced by $(f_{\mathcal{A},\alpha_0}^{(1)})^{-1}(x)$.

[5]There is a misprint in equation (2.2.27) of Proposition 2.2.5 in [H3]. The first term in the right-hand side of (2.2.27) should be $-\alpha_0^{-j}\mathcal{B}_j$.

A *superalgebra* is an (associative) algebra A (with identity $1 \in A$), such that

(i) A is a $\mathbb{Z}_2$-graded algebra
(ii) $ab = (-1)^{\eta(a)\eta(b)} ba$ for a, b homogeneous in A.

For example, the exterior (or Grassmann) algebra $\bigwedge(V)$ over a vector space V is naturally a superalgebra.

A $\mathbb{Z}_2$-graded vector space $\mathfrak{q}$ is said to be a *Lie superalgebra* if it has a bilinear operation $[\cdot, \cdot]$ such that for u, v homogeneous in $\mathfrak{q}$,

$$
\begin{aligned}
(i) \quad & [u, v] \in \mathfrak{q}^{(\eta(u)+\eta(v)) \bmod 2} \\
(ii) \quad & [u, v] = -(-1)^{\eta(u)\eta(v)} [v, u] && \text{(skew-symmetry)} \\
(iii) \quad & (-1)^{\eta(u)\eta(w)} [[u, v], w] + (-1)^{\eta(v)\eta(u)} [[v, w], u] \\
& \qquad + (-1)^{\eta(w)\eta(v)} [[w, u], v] = 0. && \text{(Jacobi identity)}
\end{aligned}
$$

Remark 4.6 Given a Lie superalgebra $\mathfrak{q}$ and a superalgebra A, $(A^0 \otimes \mathfrak{q}^0) \oplus (A^1 \otimes \mathfrak{q}^1)$ is a Lie algebra with bracket given by

$$[au, bv] = (-1)^{\eta(b)\eta(u)} ab[u, v] \tag{79}$$

(with obvious notation), where we have suppressed the tensor product symbol. Note that the bracket on the left-hand side of (79) is a Lie algebra bracket, and the bracket on the right-hand side is a Lie superalgebra bracket. If $A = \bigwedge(V)$ for some vector space V, then this Lie algebra is called the *Grassmann envelope of the Lie superalgebra* $\mathfrak{q}$ *associated with* A.

Consider a Lie superalgebra $\mathfrak{q}$ that also has a compatible $\mathbb{Z}$-grading. Fix a Grassmann algebra $A = \bigwedge(V)$. We can now apply Corollary 4.1 to the Grassmann envelope of $\mathfrak{q}$ associated with A. In addition, we can apply Corollary 4.1 to the Grassmann envelope of a $\frac{1}{T}\mathbb{Z}$-graded Lie superalgebra associated with A by regrading.

Application 4.7 *The $N = 1$ Neveu-Schwarz algebra.* The $N = 1$ Neveu-Schwarz Lie superalgebra, $\mathfrak{ns}$, is a superextension of the Virasoro algebra. Thus $\mathfrak{ns}^0$ is the Virasoro algebra $\mathfrak{v}$ as in Application 4.3, and for $j \in \mathbb{Z} + \frac{1}{2}$,

we have $\mathfrak{ns}_j = \mathbb{C}G_j$. The remaining supercommutation relations are

$$\begin{aligned}
\left[G_{m+\frac{1}{2}}, L_n\right] &= \left(m - \frac{n-1}{2}\right) G_{m+n+\frac{1}{2}} \\
\left[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}\right] &= 2L_{m+n} + \frac{1}{3}(m^2+m)\delta_{m+n,0}c \\
\left[G_{m+\frac{1}{2}}, c\right] &= 0
\end{aligned}$$

for $m, n \in \mathbb{Z}$.

Take $\mathfrak{p} = (A^0 \otimes \mathfrak{ns}^0) \oplus (A^1 \otimes \mathfrak{ns}^1)$, $p_j = L_j$ for $j \in \mathbb{Z} \setminus \{0\}$, and $p_{j-\frac{1}{2}} = a_{j-\frac{1}{2}} G_{j-\frac{1}{2}}$ for $j \in \mathbb{Z}$ where $a_{j-\frac{1}{2}} \in A^1$. Let $\mathcal{A}_j$ and $\mathcal{B}_{-j}$ be commuting formal variables for $j \in \frac{1}{2}\mathbb{Z}$, $j > 0$, and set

$$\begin{aligned}
g^+ &= \sum_{j \in \mathbb{Z}_+} \left(\mathcal{A}_j L_j + \mathcal{A}_{j-\frac{1}{2}} a_{j-\frac{1}{2}} G_{j-\frac{1}{2}}\right) \\
g^- &= \sum_{j \in -\mathbb{Z}_+} \left(\mathcal{B}_j L_j + \mathcal{B}_{j+\frac{1}{2}} a_{j+\frac{1}{2}} G_{j+\frac{1}{2}}\right)
\end{aligned}$$

($\mathbb{Z}_+$ denoting the positive integers). By Corollary 4.1, there exist unique $\Psi^- \in \mathfrak{p}^-[[\mathcal{A}, \mathcal{B}]]$, $\Psi^0 \in \mathfrak{p}_0[[\mathcal{A}, \mathcal{B}]]$ and $\Psi^+ \in \mathfrak{p}^+[[\mathcal{A}, \mathcal{B}]]$, satisfying (61) - (63) such that

$$e^{g^+} e^{g^-} = e^{\Psi^-} e^{\Psi^+} e^{\Psi^0}, \tag{80}$$

and formula (65) holds. Since $\{L_j, G_{j-\frac{1}{2}}\}_{j \in \mathbb{Z}} \cup \{c\}$ is a basis for the Neveu-Schwarz algebra, we can write

$$\Psi^- = \sum_{j \in -\mathbb{Z}_+} \left(\Psi_j L_j + \Psi_{j+\frac{1}{2}} G_{j+\frac{1}{2}}\right), \qquad \Psi^+ = \sum_{j \in \mathbb{Z}_+} \left(\Psi_j L_j + \Psi_{j-\frac{1}{2}} G_{j-\frac{1}{2}}\right)$$

and

$$\Psi^0 = \Psi_0 L_0 + \Gamma c$$

where $\Psi_j, \Gamma \in A[[\mathcal{A}, \mathcal{B}]]$ for $j \in \frac{1}{2}\mathbb{Z}$. Since c is central,

$$e^{\Psi^0} = \exp\left(\Psi_0 L_0 + \Gamma c\right) = \exp\left(\Psi_0 L_0\right) \exp\left(\Gamma c\right), \tag{81}$$

and for $j \in \mathbb{Z}_+$, we have

$$\Psi_{-j} = \mathcal{B}_{-j} + \sum_{m > j} \left(\mathcal{A}_{-j+m} \mathcal{B}_{-m} (-j + 2m)\right) \tag{82}$$

$$- 2\mathcal{A}_{-j+m-\frac{1}{2}}\mathcal{B}_{-m+\frac{1}{2}}a_{-j+m-\frac{1}{2}}a_{-m+\frac{1}{2}}\Big) + \mathcal{P}_{-j}(\mathcal{A},\mathcal{B}),$$

$$\Psi_{-j+\frac{1}{2}} = \mathcal{B}_{-j+\frac{1}{2}}a_{-j+\frac{1}{2}} + \sum_{m>j}\Bigg(\mathcal{A}_{-j+m}\mathcal{B}_{-m+\frac{1}{2}}a_{-m+\frac{1}{2}}\left(-\frac{j}{2}+\frac{3m}{2}-\frac{1}{2}\right) \tag{83}$$

$$+ \mathcal{A}_{-j+m-\frac{1}{2}}\mathcal{B}_{-m+1}a_{-j+m-\frac{1}{2}}\left(-j+\frac{3m}{2}-1\right)\Bigg) + \mathcal{P}_{-j+\frac{1}{2}}(\mathcal{A},\mathcal{B}),$$

$$\Psi_j = \mathcal{A}_j + \sum_{m\in\mathbb{Z}_+}\Big(\mathcal{A}_{j+m}\mathcal{B}_{-m}(j+2m) \tag{84}$$

$$- 2\mathcal{A}_{j+m-\frac{1}{2}}\mathcal{B}_{-m+\frac{1}{2}}a_{j+m-\frac{1}{2}}a_{-m+\frac{1}{2}}\Big) + \mathcal{P}_j(\mathcal{A},\mathcal{B}),$$

$$\Psi_{j-\frac{1}{2}} = \mathcal{A}_{j-\frac{1}{2}}a_{j-\frac{1}{2}} + \sum_{m\in\mathbb{Z}_+}\Bigg(\mathcal{A}_{j+m-1}\mathcal{B}_{-m+\frac{1}{2}}a_{-m+\frac{1}{2}}\left(\frac{j}{2}+\frac{3m}{2}-1\right) \tag{85}$$

$$+ \mathcal{A}_{j+m-\frac{1}{2}}\mathcal{B}_{-m}a_{j+m-\frac{1}{2}}\left(j+\frac{3m}{2}-\frac{1}{2}\right)\Bigg) + \mathcal{P}_{j-\frac{1}{2}}(\mathcal{A},\mathcal{B}),$$

$$\Psi_0 = \sum_{m\in\mathbb{Z}_+}\Big(2\mathcal{A}_m\mathcal{B}_{-m}m - 2\mathcal{A}_{m-\frac{1}{2}}\mathcal{B}_{-m+\frac{1}{2}}a_{m-\frac{1}{2}}a_{-m+\frac{1}{2}}\Big) \tag{86}$$

$$+ \mathcal{P}_0(\mathcal{A},\mathcal{B}),$$

$$\Gamma = \sum_{m\in\mathbb{Z}_+}\Bigg(\mathcal{A}_m\mathcal{B}_{-m}\frac{(m^3-m)}{12} \tag{87}$$

$$- \mathcal{A}_{m-\frac{1}{2}}\mathcal{B}_{-m+\frac{1}{2}}a_{m-\frac{1}{2}}a_{-m+\frac{1}{2}}\frac{(m^2-m)}{3}\Bigg) + \Gamma_0(\mathcal{A},\mathcal{B}),$$

where $\mathcal{P}_l(\mathcal{A},\mathcal{B})$, $\Gamma_0(\mathcal{A},\mathcal{B}) \in A[[\mathcal{A},\mathcal{B}]]$, for $l \in \frac{1}{2}\mathbb{Z}$, contain only terms of total order three or more in the $\mathcal{A}_m$'s and $\mathcal{B}_m$'s with order at least one in the $\mathcal{A}_m$'s and at least one in the $\mathcal{B}_m$'s, for $m \in \frac{1}{2}\mathbb{Z}$.

Remark 4.8 (cf. Remark 4.4) In $N = 1$ superconformal field theory, equation (80) corresponds to calculating the uniformizing function to obtain a canonical supersphere with tubes from the sewing together of two canonical superspheres with tubes in the moduli space of superspheres with tubes under global superconformal equivalence. This moduli space along with the sewing operation is the supergeometric structure underlying a supergeometric vertex operator superalgebra [B1], [B2]. Equation (80) also corresponds to a certain change of variables and "normal ordering" of the operators $L_j, G_{j-\frac{1}{2}}$, $j \in \mathbb{Z}$, generated by the Neveu-Schwarz element in an algebraic $N = 1$ vertex operator superalgebra, where by "normal ordering" we mean ordering the

operators $L_j, G_{j-\frac{1}{2}}$ so as to first act by the operators $L_j, G_{j-\frac{1}{2}}$ for $j > 0$ and then act by the operators $L_j, G_{j+\frac{1}{2}}$ for $j < 0$. The correspondence between these two procedures, one geometric and the other algebraic, is necessary for the proof of the isomorphism between the category of $N = 1$ vertex operator superalgebras and the category of $N = 1$ supergeometric vertex operator superalgebras [B1].

Remark 4.9 (cf. Remark 4.5) The results about the formal series $\Psi_j, \Gamma \in A[[\mathcal{A}, \mathcal{B}]]$, for $j \in \frac{1}{2}\mathbb{Z}$, given in equations (82) - (87) above—the explicit results about the lowest order terms and the qualitative information about the higher order terms—are exactly the results necessary for the proof of the isomorphism between the category of $N = 1$ vertex operator superalgebras and the category of $N = 1$ supergeometric vertex operator superalgebras [B1]. Equivalent results were proved by Barron in Theorem 2.3.4[6], Proposition 2.3.6, Proposition 2.6.1 and Corollary 2.6.2 in [B1]. However, the quantitative information in equations (82) - (87) is in a slightly different form than the equivalent information given in [B1] and [B2]. In [B1] and [B2] "odd" formal variables $M_{j-\frac{1}{2}}$ (resp., $N_{-j-\frac{1}{2}}$) are used instead of the composite expressions $\mathcal{A}_{j-\frac{1}{2}} a_{j-\frac{1}{2}}$ (resp., $\mathcal{B}_{j+\frac{1}{2}} a_{j+\frac{1}{2}}$) found in equations (82) - (87) and consisting of an even formal variable and an odd Grassmann variable. The "odd" formal variables $M_{j-\frac{1}{2}}$ (resp., $N_{-j+\frac{1}{2}}$) carry the same information as the corresponding composite expressions in the present work and are "odd" in the sense that they anticommute with each other and odd elements of $\mathfrak{ns}$ and commute with even formal variables and even elements of $\mathfrak{ns}$. After taking into consideration this notational change, one can see that the quantitative information in equations (82) - (87) is much more explicit than the equivalent information given in [B1]. The main difference between equations (82) - (87) and the analogous results first proved in [B1] is that in (82) - (87), the terms of total order two in the $\mathcal{A}_j$'s and $\mathcal{B}_j$'s are given explicitly, while in [B1] this information for the Ψ_j's and Γ is presented in (the corrected forms of) equations

[6]There is a misprint in formulas (2.49) and (2.50) of Theorem 2.3.4 in [B1]. In the lowest order terms for $\bar{F}^{(1)}$, the first two terms of the even part and the first term and last two terms of the odd part of the right-hand side of formula (2.49) should be replaced by $\varphi \bar{F}^{(1)}(x, \varphi)\big|_{(\mathcal{A},\mathcal{M})=\mathbf{0}}$ given in equation (2.45) of the theorem. And in the lowest order terms for $\bar{F}^{(2)}$, the first two terms of the even part and the first three terms of the odd part of the right-hand side of formula (2.50) should be replaced by $\varphi \bar{F}^{(2)}(x, \varphi)\big|_{(\mathcal{B},\mathcal{N})=\mathbf{0}}$ given in equation (2.48) of the theorem.

(2.49) and (2.50) (see footnote 6) using a representation of the $N = 1$ Neveu-Schwarz algebra in terms of superderivations in $\mathrm{End}(\mathbb{C}[x, x^{-1}][\varphi])$ with $c = 0$, where x is a formal (commuting) variable and φ is a formal anticommuting variable (see Proposition 2.4.1 in [B1]). In order to recover equations (82) - (87) above from the results in [B1], one must perform several operations, pick out coefficients, and then use Proposition 2.3.4 and Corollary 2.6.2 in [B1], allowing one to lift from the particular representation to the algebra. For example, a shorter and more straightforward proof of Proposition 3.11.1 in [B1] than that originally given can be obtained using equations (82) - (87) above. This proposition states that the supermeromorphic tangent space of the moduli space of superspheres with one incoming tube and one outgoing tube carries the structure of an $N = 1$ Neveu-Schwarz algebra with central charge zero.

Application 4.10 *The Neveu-Schwarz algebras for $N > 1$.* For Grassmann envelopes of other superextensions of the Virasoro algebra, such as the $N = 2$ Neveu-Schwarz algebra and Neveu-Schwarz algebras for higher N, the results of Corollary 4.1 similarly apply. These results have significance for the corresponding superconformal field theories and vertex operator superalgebras.

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